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PACKING RELATED PARAMETERS OF GENERALIZED JAHANGIR GRAPHS

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Abstract: In a graph $G = (V, E)$, a set $S \subseteq V(G)$ is 2-packing if $N[u] \cap N[v] = \phi$ for every $u, v \in S$, and S is called open packing if $N(u) \cap N(v) = \phi$ for every $u, v \in S$. An open packing set S is an outer-connected open packing set if either $S = V(G)$ or $\langle V - S \rangle$ is connected. The largest cardinalities of 2-packing, open packing, and outer-connected open packing in G are respectively called the 2-packing number (ρ), the open packing number (ρ^o), and the outer-connected open packing number (ρ^o_{oc}) of a graph G. In this paper, we compute these numbers for the generalized Jahangir graphs.

Keywords and Phrases: Packing number, 2-packing number, open packing number, outer-connected open packing number, Jahangir graph, generalized Jahangir graph.

2020 Mathematics Subject Classification: 05C70.

1. Introduction

Throughout this paper, we use only finite, simple, non-trivial and connected graphs, and we follow the notations and definitions mentioned in [4].

A set $S \subseteq V(G)$ is called *2-packing* of G if any pair of vertices in S having the distance of at least 3 in G and a set $S \subseteq V(G)$ is called *open packing* of G if $V(G)$ has no vertex which is a common neighbor between any two vertices in S. The *2-packing number* and the *open packing number* of G are defined as the maximum cardinalities of a maximal 2-packing and a maximal open packing of G respectively and their respective notations are denoted by ρ and ρ^o. Results regarding the 2-packing number can be found in [3, 7, 5, 14]. The open packing number was introduced in [7] and further investigations about ρ^o were done in [6, 10, 11, 12] and elsewhere. An open packing set S is called an *outer-connected open packing set(ocop-set)* if either $S = V(G)$ or $\langle V - S\rangle$ is connected. The maximum cardinality of a maximal outer-connected open packing is called the *outer-connected open packing number*, which is denoted by $\rho^o_{oc}(G)$. The concept of outer-connected open packing was initiated in [13].

For $n \geq 2$ and $m \geq 2$, the *generalized Jahangir graph* $J_{n,m}$ is a graph on $nm+1$ vertices consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}. The generalized Jahangir graph $J_{n,m}$ was introduced in [2]. Hereafter, several authors have worked on the generalized Jahangir graph for determining the covering number and domination related parameters of $J_{n,m}$, one can see [1, 8, 9]. From this motivation, we give the exact values of packing related parameters such as the 2-packing number, the open packing number and the outer-connected open packing number of $J_{n,m}$ for all $n \geq 2$ and $m \geq 3$ in this paper. To accomplish this work, we consider the following note, propositions and theorems.

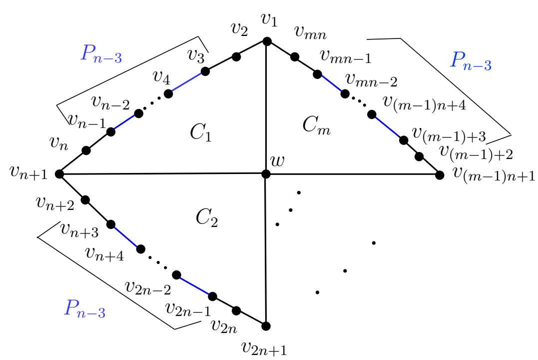

Figure 1.1. Generalized Jahangir graph $J_{n,m}$

Note 1.1. *For $J_{n,m}$, we follow the notations given in Figure 1.1. and we use the set $V(C_j^*)$ instead of the set $V(C_j) \setminus N[w]$ for all $1 \leq j \leq m$.*

Proposition 1.2. [6] *For the cycle C_n on $n \geq 3$ vertices,*

$$\rho^o(C_n) = \begin{cases} \frac{n}{2} - 1 & if\ n \equiv 2(mod\ 4) \\ \lfloor \frac{n}{2} \rfloor & otherwise \end{cases}$$

Proposition 1.3. [6] *For the path P_n on $n \geq 2$ vertices,*

$$\rho^o(P_n) = \begin{cases} \frac{n}{2} & if\ n \equiv 0(mod\ 4) \\ \lfloor \frac{n+2}{2} \rfloor & otherwise \end{cases}$$

Theorem 1.4. [7] *For any graph G, $\rho(G) \leq \gamma(G)$.*

Theorem 1.5. [10] *For any graph G, $\rho^o(G) \leq \gamma_t(G)$.*

Theorem 1.6. [9] *For $n, m \geq 2$, $\gamma(J_{n,m}) = \lceil \frac{mn}{3} \rceil$.*

Theorem 1.7. [9] *For $m \geq 2$, $\gamma_t(J_{3,m}) = m + 1$.*

Theorem 1.8. [9] *For $n \geq 4$ and $m \geq 2$,*

$$\gamma_t(J_{n,m}) = \begin{cases} \frac{nm}{2} & if\ nm \equiv 0(mod\ 4) \\ \frac{nm}{2} + 1 & if\ nm \equiv 2(mod\ 4) \\ \frac{nm+1}{2} & Otherwise \end{cases}$$

Theorem 1.9. [13] *For any graph G, $\rho^o_{oc}(G) \leq \rho^o(G)$.*

2. Main Results

In this section, we evaluate the exact values of ρ, ρ^o and ρ^o_{oc} for $J_{n,m}$. Let us start with the following observations.

Observation 2.1. *Let $S \subseteq V(J_{n,m})$. Then*

(i) $|S \cap N(w)| \leq 1$ *if S is an open packing set of $J_{n,m}$.*

(ii) $|S \cap N[w]| \leq 1$ *if S is a 2-packing set of $J_{n,m}$.*

Observation 2.2. *Let S be an open packing set of $J_{n,m}$ and let $v \in S$. Then for any $x \in N(v)$, the set S contains no vertex from $N(x) \setminus \{v\}$.*

The following theorem gives the value of the 2-packing number of $J_{n,m}$

Theorem 2.3. *For the generalized Jahangir graph $J_{n,m}$ with $m \geq 3$ and $n \geq 2$, we have*

$$\rho(J_{n,m}) = \begin{cases} \frac{mn}{3} & if\ n \equiv 0(mod\ 3) \\ \left\lfloor \frac{n}{3} \right\rfloor m + 1 & if\ n \equiv 1(mod\ 3) \\ \left\lceil \frac{m(2n-1)}{6} \right\rceil & if\ n \equiv 2(mod\ 3) \end{cases}$$

Proof. First assume that $n = 2$ and $m \geq 3$ and let $S = \{v_3\} \cup \{v_{2+4j} : 1 \leq j \leq \lceil \frac{m}{2} \rceil - 1\}$. Then S is a 2-packing set of $J_{2,m}$ as the distance between any two vertices of S is at least 3 in $J_{2,m}$ so that $\rho(J_{2,m}) \geq |S| = 1 + \left\lceil \frac{m}{2} \right\rceil - 1 = \left\lceil \frac{m}{2} \right\rceil$. For the other inequality, let D be a maximal 2-packing set of $J_{2,m}$. Then the vertex w of $J_{2,m}$ is either belongs to D or not belongs to D. If $w \in D$, then $D = \{w\}$. Suppose that $w \notin D$. Then by Observation 2.1.(ii), D contains at most one vertex from $N(w)$ and at most $\left\lceil \frac{m}{2} \right\rceil - 1$ vertices from $V(J_{2,m}) \setminus N[w]$ and hence $\rho(J_{2,m}) \leq |D| \leq 1 + \lceil \frac{m}{2} \rceil - 1$.

Now, we consider the graph $J_{n,m}$ with $m \geq 3$ and $n \geq 3$.

If $n \equiv 0(mod\ 3)$, then the set $S_1 = \left\{v_{3a-1} : 1 \leq a \leq \frac{mn}{3}\right\}$ is a 2-packing set of $J_{n,m}$ and hence $\rho(J_{n,m}) \geq |S_1| = \frac{mn}{3}$ and by Theorems 1.4 and 1.6, we obtain the inequality $\rho(J_{n,m}) \leq \gamma(J_{n,m}) = \lceil \frac{mn}{3} \rceil$ and hence $\rho(J_{n,m}) = \frac{mn}{3}$. For otherwise consider the following cases.

Case 1. $n \equiv 1(mod\ 3)$

Consider the set $Q = \cup_{i=0}^{m-1} Q_i \cup \{w\}$, where $Q_i = \left\{v_{in+3a} : 1 \leq a \leq \left\lfloor \frac{n}{3} \right\rfloor\right\}$ for all $0 \leq i \leq m-1$. Since the distance between any two vertices in the set Q is more than 2, the set Q is a 2-packing set of $J_{n,m}$ and so $\rho(J_{n,m}) \geq |Q| = 1 + m \left\lfloor \frac{n}{3} \right\rfloor$. Since by Observation 2.1.(ii), any maximal 2-packing set of $J_{n,m}$ contains at most one vertex from $N[w]$ and at most $\left\lfloor \frac{n}{3} \right\rfloor$ vertices from set $V(C_j^*)$ for each j, $1 \leq j \leq m$, it follows that $\rho(J_{n,m}) \leq 1 + m \left\lfloor \frac{n}{3} \right\rfloor$ and therefore $\rho(J_{n,m}) = \left\lfloor \frac{n}{3} \right\rfloor m + 1$.

Case 2. $n \equiv 2(mod\ 3)$

Define $R_1 = \left\{v_{3a+1} : 0 \leq a \leq \frac{n-2}{3}\right\}$, $R_m = \left\{v_{(m-1)n+3b+3} : 0 \leq b \leq \frac{n-5}{3}\right\}$ and

$$R_j = \begin{cases} v_{jn+3c+2} & if\ j\ is\ even \\ v_{jn+3d+3} & if\ j\ is\ odd \end{cases}$$

for all $2 \leq j \leq m-1$, where $1 \leq c \leq \frac{n+1}{3}$ and $1 \leq d \leq \frac{n-2}{3}$. Let $R = \cup_{j=1}^{m} R_j$. Then any two vertices in R are at distance at least three, which implies that R is a 2-packing set of $J_{n,m}$ and hence $\rho(J_{n,m}) \geq |R| = \left(\frac{n-2}{3} + 1\right) + \left(\frac{n-5}{3} + 1\right) + \left(\frac{m-2}{2}\left[\frac{n-2}{3} + \frac{n+1}{3}\right]\right) = \frac{m(2n-1)}{6}$. Let D_1 be a maximal 2-packing set of $J_{n,m}$. If

$w \in D_1$, then D_1 contains at most $\left(\frac{n-2}{3}\right)$ vertices from each set $V(C_x^*)$, where $1 \leq x \leq m$ and thus $\rho(J_{n,m}) \leq |D_1| = m\left(\frac{n-2}{3}\right)$. For otherwise let $|D_1 \cap N[w]| \leq 1$ which is guaranteed by Observation 2.1.(ii). Suppose that $|D_1 \cap N[w]| = 1$. Without loss of generality, let $D_1 \cap N[w] = \{v_1\}$. Then D_1 has at most $\left(\frac{n-2}{3}\right)$ vertices from each set $V(C_1^*)$, $V(C_m^*)$ and $V(C_y^*)$, where y is odd $(2 \leq y \leq m-1)$ together with at most $\left(\frac{n+1}{3}\right)$ vertices from $V(C_z^*)$, where z is even $(2 \leq z \leq m-1)$, which implies that $\rho(J_{n,m}) \leq |D_1| = 1 + \left(\frac{m+2}{2}\right)\left(\frac{n-2}{3}\right) + \left(\frac{m-2}{2}\right)\left(\frac{n+1}{3}\right) = \frac{m(2n-1)}{6}$. If $D_1 \cap N[w] = \phi$, then all the vertices in D_1 are from $V(C_r^*)$, where $1 \leq r \leq m$. In particular, D_1 contains $\left(\frac{n-2}{3}\right)$ vertices for odd r and $\left(\frac{n+1}{3}\right)$ vertices for even r. Therefore, $\rho(J_{n,m}) \leq \left(\frac{m-1}{2}\right)\left(\frac{n-2}{3}\right) + \left(\frac{m+1}{2}\right)\left(\frac{n+1}{3}\right) = \frac{2m(n-1)+3}{6}$.

Next we determine the value of the open packing number for generalized Jahangir graph.

Theorem 2.4. *Let $J_{n,m}$ be the generalized Jahangir graph with $n \geq 2$ and $m \geq 3$. Then*

$$\rho^o(J_{n,m}) = \begin{cases} \frac{mn}{2} & if\ n \equiv 0(mod\ 4) \\ \left\lfloor \frac{m(n-1)}{2} + 1 \right\rfloor & otherwise \end{cases}$$

Proof. For $n = 2$ and $m \geq 3$, the set $S = \{v_1, v_2\} \cup \left\{v_{2+4j} : 1 \leq j \leq \left\lfloor \frac{m+2}{2} \right\rfloor - 2\right\}$ is an open packing set of $J_{2,m}$ because of the intersection of open neighborhood of any two vertices in S is empty. Therefore, $\rho^o(J_{2,m}) \geq |S| = 2 + \left\lfloor \frac{m+2}{2} \right\rfloor - 2$. Now, take D as a maximal open packing of $J_{2,m}$. If D contains the vertex w, then by the Observation 2.1.(i), D contains at most one vertex from $N(w)$ and thus $\rho^o(G) \leq |D| \leq 2$. Suppose $w \notin D$. Then D contains exactly one vertex from $N(w)$ and at most $\frac{m}{2}$ vertices from $V(J_{2,m}) \setminus N[w]$ and hence $\rho^o(J_{2,m}) \leq |D| \leq \frac{m}{2} + 1 = \frac{m+2}{2}$.
Now, let the graph $J_{n,m}$ with $n \geq 3$ and $m \geq 3$. Suppose that $n \equiv 0(mod\ 4)$. Then the set $B_1 = \left\{v_{2+4j}, v_{3+4j} : 0 \leq j \leq \frac{nm}{4} - 1\right\}$ forms an open packing set of $J_{n,m}$ so that $\rho^o(J_{n,m}) \geq |B_1| = \frac{mn}{2}$ and by Theorems 1.5 and 1.8, we have $\rho^o(J_{n,m}) \leq \gamma_t(J_{n,m}) = \frac{mn}{2}$. Therefore $\rho^o(J_{n,m}) = \frac{mn}{2}$ when $n \equiv 0(mod\ 4)$. If $n \not\equiv 0(mod\ 4)$, then consider the following cases.

Case 1. $n \equiv 1(mod\ 4)$

The set $B_2 = \cup_{i=1}^{m} S_i \cup \{w\}$, where $S_i = \left\{v_{(i-1)n+4a-1}, v_{(i-1)n+4a} : 1 \leq a \leq \frac{n-1}{4}\right\}$ is an open packing set of $J_{n,m}$ so that $\rho^o(J_{n,m}) \geq |B_2| = \frac{m(n-1)}{2} + 1$. Let us take a maximal open packing of $J_{n,m}$ be D_1.

Subcase 1.1. $w \in D_1$

Suppose that $D_1 \cap N(w) = \phi$. Since by Observation 2.2, D_1 contains the vertices only from m distinct paths P_{n-3} of $J_{n,m}$ mentioned in Figure 1.1. Now, by Proposition 1.3, D_1 has at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ vertices from each m paths in $J_{n,m}$ and thus $\rho^o(J_{n,m}) \leq |D_1| \leq 1 + \frac{m(n-1)}{2}$. Suppose $D_1 \cap N(w) \neq \phi$. Then by Observation 2.1.(i), $|D_1 \cap N(w)| = 1$. Without loss of generality, let $D_1 \cap N(w) = \{v_1\}$. Then D_1 can have at most $\frac{n-3}{2}$ vertices from each set $V(C_1^*)$ and $V(C_m^*)$. Furthermore, D_1 has at most $\frac{n-1}{2}$ vertices from each set $V(C_j^*)$, $2 \leq j \leq m-1$, which implies that $\rho^o(J_{n,m}) \leq |D_1| \leq 2 + (m-2)\left(\frac{n-2}{2}\right) + 2\left(\frac{n-3}{2}\right) = \frac{m(n-2)}{2} + 1$.

Subcase 1.2. $w \notin D_1$

The set D_1 contains at most $\frac{n-1}{2}$ vertices from each set $V(C_j^*)$, $1 \leq j \leq m$ and exactly one vertex from $N(w)$ so that $\rho^o(J_{n,m}) \leq |D_1| = \frac{m(n-1)}{2} + 1$.

Case 2. $n \equiv 2(mod\ 4)$

For $1 \leq j \leq m$, define the sets Q_j as follows. Let $Q_1 = \{v_1, v_2\} \cup \{v_{4b+1}, v_{4b+2}\}$ and for $2 \leq j \leq m$, let $Q_j = \left\{v_{(j-1)n+4b-1}, v_{(j-1)n+4b}\right\}$ if j is even and let $Q_j = \left\{v_{(j-1)n+4b-2}, v_{(j-1)n+4b-1}\right\}$ if j is odd, where $1 \leq b \leq \frac{n-2}{4}$. Now, consider the set $B_3 = \cup_{j=1}^{m} Q_j \cup \{v_{zn} : z\ is\ odd\}$, where $3 \leq z \leq m-1$ when m is even and $3 \leq z \leq m-2$ when m is odd. Since no two vertices in B_3 have a common vertex in $J_{n,m}$ and thus the set B_3 is an open packing set of $J_{n,m}$. Hence if m is even, then $\rho^o(J_{n,m}) \geq |B_3| = 2 + \frac{n-2}{2} + (m-1)\left(\frac{n-2}{2}\right) + \frac{m-2}{2} = \frac{m(n-1)}{2} + 1$ and if m if odd, then $\rho^o(J_{n,m}) \geq |B_3| = 2 + \frac{n-2}{2} + (m-1)\left(\frac{n-2}{2}\right) + \frac{m-3}{2} = \frac{m(n-1)+1}{2}$, which implies that $\rho^o(J_{n,m}) \geq \left\lfloor \frac{m(n-1)}{2} + 1 \right\rfloor$. For the other inequality, let D_2 be a maximal open packing set of $J_{n,m}$. If $w \in D_2$, then by similar argument in Subcase 1.1, we have $\rho^o(J_{n,m}) \leq |D_2| = \frac{m(n-1)}{2} + 1$. Suppose $w \notin D_2$. Then $|N(w) \cap D_2| = 1$. Then D_2 has at most $\frac{n-2}{2}$ vertices from $V(C_m^*)$ and for $1 \leq i \leq m-1$, the set D_2 has at most $\frac{n}{2}$ vertices from each set $V(C_i^*)$, (i is odd) and at most $\frac{n-2}{2}$ vertices from each $V(C_i^*)$, (i is even) in $J_{n,m}$, it follows that $\rho^o(J_{n,m}) \leq |D_2| = \left\lfloor \frac{m(n-1)}{2} + 1 \right\rfloor$.

Case 3. $n \equiv 3(mod\ 4)$

Consider the set $B_4 = \cup_{k=3}^{m} R_k \cup R_1 \cup R_2 \cup A_t$, where $R_k = \left\{v_{(k-1)n+4c-1}, v_{(k-1)n+4c}\right\}$, $R_1 = \{v_1, v_2\} \cup \{v_{4c+1}, v_{4c+2}\}$, $R_2 = \{v_{n+4d-2}, v_{n+4d-1}\}$, $A_t = \{v_{tn} : 3 \leq t \leq m-1\}$, $1 \leq c \leq \frac{n-3}{4}$ and $1 \leq d \leq \frac{n+1}{4}$. Therefore $|\cup_{k=3}^{m} R_k| = \sum_{k=3}^{m} |R_k| = (m-2)\left(\frac{n-3}{2}\right)$, $|R_1| = 2 + \frac{n-3}{2}$, $|R_2| = \frac{n+1}{2}$ and $|A_t| = m-3$. Since no two vertices in B_4 have a common neighbor so that B_4 forms an open packing set of $J_{n,m}$ and hence $\rho^o(J_{n,m}) \geq |B_4| = \frac{m(n-1)}{2} + 1$. Now, take D_3 as a maximal open packing of $J_{n,m}$.

Subcase 3.1. $w \in D_3$

Suppose $D_3 \cap N(w) = \phi$. Then by Observation 2.2.(i), $|D_3| = m\rho^o(P_{n-3}) + 1$ and by Proposition 1.3, $\rho^o(P_{n-3}) = \frac{n-3}{2}$ so that $\rho^o(J_{n,m}) \leq |D_3| = 1 + \frac{m(n-3)}{2}$. If $D_3 \cap N(w) \neq \phi$, then by Observation 2.1.(i), $|D_3 \cap N(w)| = 1$ and let $D_3 \cap N(w) = \{v_1\}$. Since $v_1 \in D_3$ and by Observation 2.2, the vertices v_3 and v_{mn-1} does not belong to D_3. Thus D_3 can have vertices from each two paths P_{n-4} placed on C_1 and C_m and from remaining $(m-2)$ paths P_{n-3} (mentioned in Figure 1.2.) in $J_{n,m}$. It follows that $|D_3| = 2\rho^o(P_{n-4}) + (m-2)\rho^o(P_{n-3}) + 2$ and by Proposition 1.3, $|D_3| = 2\left\lfloor\frac{n-2}{2}\right\rfloor + (m-2)\left(\frac{n-3}{2}\right) + 2$. Hence $\rho^o(J_{n,m}) \leq |D_3|$

Subcase 3.2. $w \notin D_3$

In this case D_3 contains exactly one vertex from $N(w)$ and D_3 contains at most $\frac{n+1}{2}$ vertices from $V(C_1^*)$ and at most $\frac{n-3}{2}$ vertices from $V(C_m^*)$. Moreover D_3 contain at most $\frac{n-1}{2}$ each set $V(C_j^*)$, $2 \leq j \leq m-1$ so that $\rho^o(J_{n,m}) \leq |D_3| = 1 + \frac{m(n-1)}{2}$. This completes the proof.

The following theorem gives the exact value of an outer-connected open packing set for the Jahangir graph $J_{n,m}$.

Theorem 2.5. *For a generalized Jahangir graph $J_{n,m}$ with $n \geq 2$ and $m \geq 3$, we have*

$$\rho^o_{oc}(J_{n,m}) = \begin{cases} \left\lceil\frac{m-1}{2}\right\rceil + 1 & if\ n = 2 \\ m+1 & if\ n = 3 \\ 2m & if\ n \geq 4 \end{cases}$$

Proof. Let $n = 2$ and $m \geq 3$. Then the set $S = \{v_1\} \cup \left\{v_{4i+2} : 0 \leq i \leq \left\lceil\frac{m-1}{2}\right\rceil - 1\right\}$ is open packing and $\langle V(J_{2,m}) \setminus S\rangle$ is connected and hence S is an ocop-set of $J_{2,m}$ so that $\rho^o_{oc}(J_{2,m}) \geq \left\lceil\frac{m-1}{2}\right\rceil + 1$. Now, let D be a maximal ocop-set of $J_{2,m}$. Then D should contains exactly one vertex in $N(w)$, let it be x. Furthermore, if $w \in D$ then $\langle J_{2,m} \setminus \{w, x\}\rangle$ is isomorphic to the path P_{nm-1}. Since each internal vertex of a path is a cut vertex, it follows that D does not contains any internal vertex of P_{nm-1} and by Observation 2.2, end vertices of P_{nm-1} does not belong to D and so $D = \{w, x\}$ is the one and only ocop-set of $J_{2,m}$. If $w \notin D$, then D has exactly one vertex from each cycle C_i where $1 \leq i \leq m-1$ and i is odd, which implies that $|D| \leq \left\lceil\frac{m-1}{2}\right\rceil + 1$ and hence $\rho^o_{oc}(J_{2,m}) = \left\lceil\frac{m-1}{2}\right\rceil + 1$.

If $n = 3$ and $m \geq 3$, then the set $S_1 = \{v_1, v_2, v_5\} \cup \{v_{3j+3} : 1 \leq j \leq m-2\}$ is an ocop-set of $J_{3,m}$ and therefore $\rho^o_{oc}(J_{3,m}) \geq m+1$. Now, from the Thereoms 1.5, 1.8 and 1.9, we have $\rho^o_{oc}(G) \leq \rho^o(J_{n,m}) \leq \gamma_t(J_{n,m}) = m+1$.

Now, consider the graph $J_{n,m}$ for all $n \geq 4$ and $m \geq 3$. For $1 \leq i, j \leq 2m$, define the set $S_2 = \{v_{2i} : i \text{ is odd}\} \cup \{v_{2j-1} : j \text{ is even}\}$. Then the graph $\langle V(J_{n,m}) \setminus S_2 \rangle$ is connected and no two vertices of S_2 have a common neighbor in $J_{n,m}$, which leads that S_2 is an ocop-set of $J_{n,m}$ and hence $\rho^o_{oc}(J_{n,m}) \geq 2m$.

For the other inequality let D_1 be a maximal ocop-set of $J_{n,m}$. Suppose $w \in D_1$. Then by Observation 2.1.(i), $|D_1 \cap N(w)| \leq 1$. If $|D_1 \cap N(w)| = 1$, then $|D_1| = 2$. Suppose $|D_1 \cap N(w)| = 0$. Then by Observation 2.2 and by the definition of ocop-set, D_1 has exactly one vertex from $V(J_{n,m}) \setminus N[w]$ when $n = 4$ and D_1 has exactly one pair of adjacent vertices from $V(J_{n,m}) \setminus N[w]$ when $n \geq 5$ so that $|D_1| \leq 3$.

Now, consider the set D_1 such that $w \notin D_1$. If $D_1 \cap N(w) = \phi$, then D_1 has exactly two adjacent vertices from each set $V(C_r) \setminus N[w]$, where $1 \leq r \leq m$, and hence $|D_1| \leq 2m$; Otherwise let $D_1 \cap N(w) = \{z\}$. Then D_1 has exactly one vertex which is adjacent to z from the two consecutive cycles C_s and C_t in which the cycles C_s and C_t share the common edge wz, where $1 \leq s, t \leq m$ and $s \neq t$. Moreover, from each set $V(C_k) \setminus N[w]$, one pair of adjacent vertices belong to D_1, where $1 \leq k \leq m$ and $k \neq s, t$, which gives that $|D_1| \leq 2 + 2m - 4 = 2m - 2$. Hence in all possibilities of the set D_1, we have $\rho^o_{oc}(J_{n,m}) \leq 2m$.

3. Conclusion

In this paper, we completely determined some packing related parameters such as 2-packing number, open packing number, and outer-connected open packing number for the generalized Jahangir graph. In this way, finding the values of k-limited packing number for all $k \geq 2$ of generalized Jahangir graph $J_{n,m}$ is an interesting one.

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